2.5

Complex Zeros and the Fundamental Theorem of Algebra

<u>Definition:</u> A complex number is any number that can be written in the form *a + bi* where *a and b* are real numbers. The real number *a* is the real part, the real number *b* is the imaginary part, and *a + bi* is the standard form.

A real number *a* is the complex number *a + 0i*, so all real numbers are also complex numbers.

<u>Addition and Subtraction of Complex Numbers</u>

Sum: (*a + bi*) + (*c + di*) = (*a + c*) + (*bi + di*)

Difference: (*a + bi*) - (*c + di*) = (*a - c*) + (*bi - di*)

Additive Inverse: (*a + bi*) is -(*a + bi*) = *-a – bi*

<u>Complex Conjugate</u>:

The complex conjugate of the complex number *z = a + bi* *is* $\bar{z} = \overline{a+bi} = a - bi$.

<u>Complex Solutions of Quadratic Equations</u>:

Discriminant is $b^2 - 4ac$ of $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If $b^2 - 4ac < 0$ then the root (solution) is a complex number

If $b^2 - 4ac > 0$ then there are two real roots (solutions)

If $b^2 - 4ac = 0$ then there is one real root (solution)

If $b^2 - 4ac < 0$ then there are two complex conjugate roots

<u>Fundamental Theorem of Algebra</u>: A polynomial function of degree n has n complex zeros (real and nonreal). Some of the zeros may be repeated.

<u>Linear Factorization Theorem:</u> If $f(x)$ is a polynomial function of degree $n > 0$, then $f(x)$ has precisely n linear factors and

$$f(x) = a(x - z_1)(x - z_2)\ldots(x - z_n)$$

where a is the leading coefficient of $f(x)$ and $z_1, z_2, \ldots, z_n$ are the complex zeros of $f(x)$. The z_i are not necessarily distinct numbers; some may be repeated.

<u>Fundamental Polynomial Connections in the Complex Case:</u> The following statements about polynomial function f are equivalent if k is a complex number:

1. $x = k$ is a solution (or root) of the equation $f(x) = 0$.
2. k is a zero of the function f.
3. $x - k$ is a factor of $f(x)$.

Examples:

Write the polynomial function in standard form, and identify the zeros of the function and the x-intercepts of its graph.

a) $f(x) = (x - 2i)(x + 2i) = x^2 + 2ix - 2ix - 4i^2 = x^2 + 4$, it has two zeros at $x = 2i$ and $x = -2i$, because the zeros are not real there are no x-intercepts.

b) $f(x) = (x - 5)(x - \sqrt{2}i)(x + \sqrt{2}i) = (x - 5)(x^2 + 2) = x^3 - 5x^2 + 2x - 10$, there are three zeros at $x = 5$, $x = \sqrt{2}i$, and $x = -\sqrt{2}i$. Only $x = 5$ is a real root so there is just one x-intercept at 5.

<u>Complex Conjugate Zeros:</u>

Suppose that $f(x)$ is a polynomial function with real coefficients. If a and b are real numbers with $b \neq 0$ and ***a + bi* is a zero** of $f(x)$, **then** its complex conjugate ***a - bi* is also a zero** of $f(x)$.

Example: Finding a polynomial given zeros.

Write a polynomial function in standard form with real coefficients whose zeros include -2, 1, $1 - 2i$.

If $1 - 2i$ is a zero, then $1 + 2i$ is also a zero. So, the factors we get from the zeros are $(x + 2)(x - 1)[x - (1 - 2i)][x - (1 + 2i)]$.

Therefore, $f(x) = (x + 2)(x - 1)[x - (1 - 2i)][x - (1 + 2i)]$

$$= (x^2 + x - 2)(x - 1 + 2i)(x - 1 - 2i)$$

$$= (x^2 + x - 2)(x^2 - x - 2ix - x + 1 + 2i + 2ix - 2i - 4i^2)$$

$$= (x^2 + x - 2)(x^2 - 2x + 5) = x^4 - x^3 + x^2 + 9x - 10$$

Finding complex zeros:

The complex number i is a zero of $f(x) = x^4 - 3x^2 - 4$, find the remaining zeros of $f(x)$ and write it in its linear factorization.

Using synthetic division we can use i to show that $f(x - i) = 0$.

$$\begin{array}{r|rrrrr} i & 1 & 0 & -3 & 0 & -4 \\ & & i & -1 & -4i & 4 \\ \hline & 1 & i & -4 & -4i & 0 \end{array}$$

the complex conjugate of i is $-i$ so,

$$\begin{array}{r|rrrr} -i & 1 & i & -4 & -4i \\ & & -i & 0 & 4i \\ \hline & 1 & 0 & -4 & 0 \end{array}$$

From the last row we get $x^2 - 4$, which factors to $(x + 2)(x - 2)$.

We can now write the linear factorization as $f(x) = (x + 2)(x - 2)(x - i)(x + i)$.

Every polynomial function with real coefficients can be written as a product of linear factors and irreducible factors, each with real coefficients.

Every polynomial function of odd degree with real coefficients has at least one real zero.

See example 6 on pg. 233.